An Alternative Characterization of Efficient Group Strategy-Proof Rules in Linear Production Economies with Convex Preferences*

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Abstract

In linear production economies, Maniquet and Sprumont (1999) characterized Pareto-efficient and group strategy-proof rules. This paper shows an alternative characterization of the rules by Pareto-efficiency, strategy-proofness, and strong non-bossiness (Ritz, 1983) when the preferences are continuous, strictly monotonic, and convex.

JEL Classification: C72, D51, D52, D61, D71.

1 Introduction

This paper considers a linear production economy in which $n \geq 2$ agents consume $m \geq 2$ divisible and private goods based on a linear production function[1]. In the economies, Maniquet and Sprumont (1999) characterized Pareto-efficient and group strategy-proof rules. A rule is a function which associates a feasible allocation with a profile of agents' preferences[2]. Group strategy-proofness, which is an incentive property of rules, requires that any group of agents cannot gain by jointly untruthful preference revelations. **Pareto-efficiency**, which is an efficiency

* This paper is a product of research which was financially supported by a designated research project of the Research Institute, Momoyama Gakuin University, (2019-2020) "A Study on Bounded Rationality in Game Theory" and JSPS KAKENHI Grant Number JP19K13661. Any errors in this paper are entirely the responsibility of the author.

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1) See Moulin and Shenker (1992), Shenker (1992), Maniquet and Sprumont (1999), Leroux (2004), Kumar (2013), and Nishizaki (2018b) for the studies on production economies. Especially, Maniquet and Sprumont (1999) and Nishizaki (2018b) studied linear production economies.

2) In this paper, a rule is defied as a direct revelation mechanism associated with a social choice function. This means that a rule is equivalent to a social choice function.

Keywords: Pareto-Efficiency, Group Strategy-Proofness, Non-Bossiness, Linear Production Economy, Convex Preference

property of rules, requires that there is no allocation leading to a Pareto improvement for agents. This paper shows an alternative characterization of Maniquet and Sprumont's (1999) rules by Pareto-efficiency, strategy-proofness, and strong non-bossiness (Ritz, 1983). **Strategy-proofness,** which is in general weaker than group strategy-proofness, requires that truthful preference revelation is a weakly dominant strategy for each agent. **Strong non-bossiness,** which is in general stronger than non-bossiness (Satterthwaite and Sonnenschein, 1981), requires that each agent cannot change the allocation by the agent's preference revelation while maintaining the agent's utility[3]. As byproduct of the alternative characterization, this paper also shows that the combination of strategy-proofness and strong non-bossiness is equivalent to group strategy-proofness under Pareto-efficient rules in the model presented here[4].

The remainder of this paper is organized as follows. Section 2 introduces the model presented here and Section 3 the properties of rules. Section 4 demonstrates the results and Section 5 concludes this paper.

2 Model

Similar to Maniquet and Sprumont (1999) and Nishizaki (2018b), this paper considers a linear production economy with $n \geq 2$ agents and $m \geq 2$ divisible and private goods. Let $I \equiv \{1, \ldots, n\}$ be the set of **agents** and $K \equiv \{1, \ldots, m\}$ be the set of **goods**. For each $i \in I$ and each $k \in K$, let $y_{ik} \in \mathbb{R}_+ \equiv \{r \in \mathbb{R} | r \geq 0\}$ be **consumption of good** $\boldsymbol{k}$ **for agent** $\boldsymbol{i}$ and $y_i \equiv (y_{ik})_{k \in K} \in \mathbb{R}^m_+$ be **consumption for agent** $\boldsymbol{i}$. Let $y \equiv (y_i)_{i \in I} \mathbb{R}^{mn}_+$ be an **allocation**. In the model presented here, a good is produced from other goods according to a technology which exhibits constant return to scale. For simplicity, let $Y \equiv \{y \in \mathbb{R}^{mn}_+ | \sum_{i \in I} \sum_{k \in K} y_{ik} \leq 1\}$ be the set of **feasible allocations**.

A preference for an agent is represented by a binary relation defined on $\mathbb{R}^m_+$. For each $i \in I$, let R_i be a **preference for agent** $\boldsymbol{i}$ and I_i be the indifference associated with R_i. This paper assumes that each preference is continuous, strictly monotonic (that is, strictly increasing in consumption of each good), and convex. For each $i \in I$, let $\mathscr{R}_i$ be the set of such preferences for agent i. Let $R \equiv (R_i)_{i \in I}$ be a profile of preferences and $\mathscr{R} \equiv \Pi_{i \in I} \mathscr{R}_i$ be the set of profiles of preferences. For each $i \in I$, let $R_{-i} \equiv (R_h)_{h \in I \setminus \{i\}}$ be a profile of preferences other than agent i and $\mathscr{R}_{-i} \equiv \Pi_{h \in I \setminus \{i\}} \mathscr{R}_h$ be the set of profiles of preferences other than agent i. In addition, for each $i, j \in I$, let $R_{-i,j} \equiv (R_h)_{h \in I \setminus \{i, j\}}$ be a profile of preferences other than agents i and j.

3) Strong non-bossiness was called non-corruptibility by Ritz (1983). See Saijo, Sjöström, and Yamato (2007), Berga and Moreno (2009), and Nishizaki (2012, 2014, 2018a) for the studies on strong non-bossiness.

4) Non-bossiness is necessary for group strategy-proofness in some environments; for example, non-excludable public good economies (Serizawa, 1994), pure exchange economies (Barberà and Jackson, 1995), the problems of cost sharing (Mutuswami, 2005), and the problems of allocating indivisible goods without monetary transfers (Pápai, 2000; Takamiya, 2001).

Remark 1. Maniquet and Sprumont (1999) imposed continuity, strict monotonicity, and some richness conditions which convexity satisfies[5]. Nishizaki (2018b) imposed "strict" convexity in addition to continuity and strict monotonicity[6].

Agents collectively choose a feasible allocation according to a rule. Let $f: \mathscr{R} \to Y$ be a **rule** which associates a feasible allocation $y \in Y$ with a profile of preferences $R \in \mathscr{R}$. For each $R \in \mathscr{R}$ and each $i \in I$, let $f_i(R)$ be the consumption for agent i at the allocation $f(R)$.

3 Properties of Rules

This paper studies Pareto-efficient, strategy-proof, and strongly non-bossy rules in the above model. Pareto-efficiency requires that there is no allocation leading to a Pareto improvement for agents. Strategy-proofness requires that truthful preference revelation is a weakly dominant strategy for each agent. Strong non-bossiness (Ritz, 1983), which is in general stronger than non-bossiness (Satterthwaite and Sonnenschein, 1981), requires that each agent cannot change the allocation by the agent's preference revelation while maintaining the agent's utility[7].

Definition 1. The rule f satisfies **Pareto-efficiency** if and only if for each $R \in \mathscr{R}$ and each $y \in Y$, if $y_i R_i f_i(R)$ for each $i \in I$, then $y_i I_i f_i(R)$ for each $i \in I$.

Definition 2. The rule f satisfies **strategy-proofness** if and only if for each $R, R' \in \mathscr{R}$ and each $i \in I$, $f_i(R_i, R'_{-i}) R_i f_i(R'_i, R'_{-i})$.

Definition 3. The rule f satisfies **strong non-bossiness** if and only if for each $R, R' \in \mathscr{R}$ and each $i \in I$, if $f_i(R_i, R'_{-i}) I_i f_i(R'_i, R'_{-i})$, then $f(R_i, R'_{-i}) = f(R'_i, R'_{-i})$.

4 Results

For each $i \in I$ and each $r \in \mathbb{R}_+$, let $B_i(r) \equiv \{y_i \in \mathbb{R}^m_+ | \sum_{k \in K} y_{ik} \leq r\}$ be the **consumption set for agent *i* at *r***. For each $i \in I$, each $R_i \in \mathscr{R}_i$, and each $r \in \mathbb{R}_+$, let

$$m(R_i, B_i(r)) \equiv \{y_i \in B_i(r) | y_i R_i y'_i \text{ for each } y'_i \in B_i(r)\}$$

be the set of most preferred consumption for agent i with R_i in the agent's consumption set $B_i(r)$. Similar to Maniquet and Sprumont (1999) and Nishizaki (2018b), we use the notation

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5) The richness conditions are called Assumptions (A1) and (A2) by Maniquet and Sprumont (1999).

6) Strict convexity satisfies Assumption (A1), but not (A2) introduced by Maniquet and Sprumont (1999), as stated by them.

7) The rule f satisfies non-bossiness if and only if for each $R, R' \in \mathscr{R}$ and each $i \in I$, if $f_i(R_i, R'_{-i}) = f_i(R'_i, R'_{-i})$, then $f(R_i, R'_{-i}) = f(R'_i, R'_{-i})$.

$m(R_i, r)$ as substitute for $m(R_i, B_i(r))$ for each $i \in I$, each $R_i \in \mathscr{R}_i$, and each $r \in \mathbb{R}_+$.

Remark 2. The set of most preferred consumption might not be a singleton because the preferences are convex, not "strictly" convex.

In the model presented here, Maniquet and Sprumont (1999, Lemma 2) showed a feature of Pareto-efficient and strategy-proof rules.

Lemma 1 (Maniquet and Sprumont, 1999). *If the rule f satisfies* ***Pareto-efficiency*** *and* ***strategy-proofness****, then for each $i \in I$, there is a_i: $\mathscr{R}_{-i} \to \mathbb{R}_+$ such that $f_i(R_i, R_{-i}) \in m(R_i, a_i(R_{-i}))$ for each $R \in \mathscr{R}$*[8].

Remark 3. For Lemma 1, we know that $\sum_{i \in I} a_i(R_{-i}) = 1$ for each $R \in \mathscr{R}$ by the feasibility of allocations and Pareto-efficiency.

Remark 4. For each $R, R' \in \mathscr{R}$ and each $i \in I$, Lemma 1 implies that $a_i(R_{-i}) = a_i(R'_{-i})$ if $f_i(R_i, R_{-i}) = f_i(R_i, R'_{-i})$ because the preferences are strictly monotonic.

In what follows, we focus on Pareto-efficient and strategy-proof rules stated in Lemma 1. Under such rules, the combination of strategy-proofness and strong non-bossiness requires that the set of most preferred consumption is a singleton.

Lemma 2. *Suppose that for each $i \in I$, there is a_i: $\mathscr{R}_{-i} \to \mathbb{R}_+$ such that $f_i(R_i, R_{-i}) \in m(R_i, a_i(R_{-i}))$ for each $R \in \mathscr{R}$. If the rule f satisfies* ***strategy-proofness*** *and* ***strong non-bossiness****, then for each $R \in \mathscr{R}$ and each $i \in I$, $m(R_i, a_i(R_{-i}))$ is a singleton.*

Proof. To the contrary, we suppose that there are $R \in \mathscr{R}$ and $i \in I$ such that $m(R_i, a_i(R_{-i}))$ is not a singleton. By supposition, we know that $f_i(R_i, R_{-i}) \in m(R_i, a_i(R_{-i}))$. These imply that there is $R'_i \in \mathscr{R}_i$ such that $f_i(R'_i, R_{-i}) \in m(R_i, a_i(R_{-i}))$ and

$$f_i(R'_i, R_{-i}) \neq f_i(R_i, R_{-i}).$$

Because the preferences are convex, we can take $R''_i \in \mathscr{R}_i$ such that

$$f_i(R'_i, R_{-i}),\ f_i(R_i, R_{-i}) \in m(R''_i, a_i(R_{-i})) \text{ and } f_i(R'_i, R_{-i}) I''_i f_i(R_i, R_{-i}).$$

Together with **strategy-proofness**, this implies that $f_i(R''_i, R_{-i}) I''_i f_i(R'_i, R_{-i})$. Together with **strong non-bossiness,** this implies $f(R''_i, R_{-i}) = f(R'_i, R_{-i})$. Similarly, we find that $f(R''_i, R_{-i}) = f(R_i, R_{-i})$. These imply that $f(R'_i, R_{-i}) = f(R_i, R_{-i})$, that is, $f_i(R'_i, R_{-i}) = f_i(R_i, R_{-i})$ and we have a contradiction. □

8) Note that the set of continuous, strictly monotonic, and convex preferences satisfies Assumption (A1) introduced by Maniquet and Sprumont (1999), as stated by them.

Remark 5. If the preferences are strictly convex, then the set of most preferred consumption is a singleton irrespective of imposing strong non-bossiness[9].

In addition, the combination of strategy-proofness and strong non-bossiness requires that the set of most preferred consumption is uniquely determined according the value of $a_i(R_{-i})$ stated in Lemma 1.

Lemma 3. *Suppose that for each $i \in I$, there is $a_i\colon \mathscr{R}_{-i} \to \mathbb{R}_+$ such that $f_i(R_i, R_{-i}) \in m(R_i, a_i(R_{-i}))$ for each $R \in \mathscr{R}$. If the rule f satisfies **strategy-proofness** and **strong non-bossiness**, then for each $R, R' \in \mathscr{R}$ and each $i \in I$, $m(R'_i, a_i(R_{-i})) = m(R_i, a_i(R_{-i}))$.*

Proof. To the contrary, we suppose that there are $R, R' \in \mathscr{R}$ and $i \in I$ such that $m(R'_i, a_i(R_{-i})) \neq m(R_i, a_i(R_{-i}))$. By supposition, we know that $f_i(R'_i, R_{-i}) \in m(R'_i, a_i(R_{-i}))$ and $f_i(R_i, R_{-i}) \in m(R_i, a_i(R_{-i}))$. Together with Lemma 2, these imply that $f_i(R'_i, R_{-i}) \neq f_i(R_i, R_{-i})$. Because the preferences are convex, we can take $R''_i \in \mathscr{R}_i$ such that $f_i(R'_i, R_{-i}), f_i(R_i, R_{-i}) \in m(R''_i, a_i(R_{-i}))$ and $f_i(R'_i, R_{-i}) I''_i f_i(R_i, R_{-i})$. By an argument similar to Lemma 2, we have a contradiction. □

Remark 6. If the preferences are strictly convex, then Lemma 3 does not hold because we cannot take R''_i stated in the proof of Lemma 3.

Remark 7. Lemma 3 can be rewritten as follows: if the rule f satisfies **Pareto-efficiency**, **strategy-proofness**, and **strong non-bossiness**, then for each $R, R' \in \mathscr{R}$ and each $i \in I$, $f_i(R'_i, R_{-i}) = f_i(R_i, R_{-i})$.

Based on the above results, this paper shows an alternative characterization of "unequal" budget free choice rules (Maniquet and Sprumont, 1999)[10].

Theorem. *The rule f satisfies **Pareto-efficiency**, **strategy-proofness**, and **strong non-bossiness** if and only if there is $(\alpha_i)_{i \in I} \in \mathbb{R}^n_+$ such that for each $i \in I$, $f_i(R_i, R_{-i}) \in m(R_i, \alpha_i)$ for each $R \in \mathscr{R}$, where $\sum_{i \in I} \alpha_i = 1$.*

Proof. Because the "if" part is straightforward, we only confirm the "only if" part. Let $i \in I$. By Lemma 1, we know that there is $a_i\colon \mathscr{R}_{-i} \to \mathbb{R}_+$ such that $f_i(R_i, R_{-i}) \in m(R_i, a_i(R_{-i}))$ for each $R \in \mathscr{R}$. Due to Remark 3, we also know that $\sum_{i \in I} a_i(R_{-i}) = 1$ for each $R \in \mathscr{R}$. Let $R \in \mathscr{R}$. Simi-

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9) This implies that strong non-bossiness is equivalent to non-bossiness used by Nishizaki (2018b) under Pareto-efficient and strategy-proof rules when the preferences are continuous, strictly monotonic, and strictly convex.

10) Maniquet and Sprumont (1999) introduced the equal budget free choice rule under which $\alpha_i = 1/n$ for each $i \in I$ in the theorem.

lar to Maniquet and Sprumont (1999, Theorem 3), this proof only shows that

$$a_i(R'_j, R_{-i,j}) = a_i(R_j, R_{-i,j}) \text{ for each } j \in I \setminus \{i\} \text{ and each } R'_j \in \mathscr{R}_j.$$

Let $j \in I \setminus \{i\}$ and $R'_j \in \mathscr{R}_j$. Due to Remark 7, we know that $f_j(R'_j, R_{-j}) = f_j(R_j, R_{-j})$. Together with **strong non-bossiness**, this implies that $f(R'_j, R_{-j}) = f(R_j, R_{-j})$, that is, $f_i(R_i, R'_j, R_{-i,j}) = f_i(R_i, R_j, R_{-i,j})$. Due to Remark 4, this means that $a_i(R'_j, R_{-i,j}) = a_i(R_j, R_{-i,j})$. □

Based on the above theorem and the result of Maniquet and Sprumont (1999, Theorem 3), we have the following corollary[11].

Corollary. *When the rule f satisfies Pareto-efficiency, f satisfies* ***strategy-proofness*** *and* ***strong non-bossiness*** *if and only if f satisfies group strategy-proofness*[12].

5 Conclusion

The above theorem crucially depends on convexity of preferences in addition to continuity and strict monotonicity due to Remark 6. In fact, Maniquet and Sprumont (1999) showed an example of Pareto-efficient, strategy-proof, and strongly non-bossy rules which has non-constant a_i for each $i \in I$ when convexity is replaced by strict convexity[13]. In the case of strictly convex preferences, a characterization of Pareto-efficient, strategy-proof, and strongly non-bossy rules is an open question.

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11) Note that the set of continuous, strictly monotonic, and convex preferences satisfies Assumptions (A1) and (A2) introduced by Maniquet and Sprumont (1999), as stated by them.

12) The rule f satisfies group strategy-proofness if and only if for each $R, R' \in \mathscr{R}$ and each $S \subseteq I$, if $f_i(R'_S, R_{-S}) R_i f_i(R_S, R_{-S})$ for each $i \in S$, then $f_i(R'_S, R_{-S}) I_i f_i(R_S, R_{-S})$ for each $i \in S$, where (R'_S, R_{-S}) is a profile of preferences at which agent $i \in S$ has R'_i and agent $j \in I \setminus S$ has R_j.

13) Nishizaki (2018b) showed that the combination of strategy-proofness and non-bossiness is equivalent to secure implementability (Saijo, Sjöström, and Yamato, 2007) which is in general stronger than dominant strategy implementability under Pareto-efficient rules when the preferences are continuous, strictly monotonic, and strictly convex.

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(Accepted on 16 November, 2020)