On Canonical Functions and Conformal Mappings

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Abstract Another formulation of the existence theorem of canonical (meromorphic) functions on open Riemann surfaces is shown. Geometrically it implies that for given integer $n \geq \max(2g, 1)$ and a point p of Riemann surface R of genus $g (0 \leq g < \infty)$ there exist a pair of conformal mappings (normalized at pole p) of R to an n-sheeted covering surface with vertical or horizontal slits respectively. Besides, a certain integral formula for locally canonical functions is obtained.

Introduction

This paper contains a certain supplement and development of our former paper [6] and actually treats with canonical differentials and functions on general open Riemann surfaces.

In section 2, §1 we introduce a notion, called modification, for square integrable differentials on open Riemann surfaces and show the existence and construction of the modification for basic those differentials. By using these results we shall prove an integral formula, Proposition 3, which is slightly general than a formula stated in [6] without proof. Section 5, §3 contains main theorem. The essential part of this existence theorem was accomplished in [4] and [6], however we shall give here another formulation showing a generalization of the classical canonical conformal mapping theorem.

The theory of canonical functions is still infancy for Riemann surfaces of infinite genus. The final paragraph exhibits some examples of those surfaces, which give a few informations concerning the canonical functions.

§1. Modification of square integrable differentials

1. **Preliminaries** First of all we recall briefly some definitions and notations. By R we denote an arbitrary open Riemann surface. For two square integrable differentials ω_1, ω_2 on R the inner product is given by

$$(\omega_1, \omega_2) = (\omega_1, \omega_2)_R = \iint_R \omega_1 \wedge {}^*\bar{\omega}_2$$

where generally $^*\omega$ stands for the conjugate differential of ω. Let $\Gamma (= \Gamma(R))$ be the Hilbert space of square integrable *real* differentials on R with above inner product. The following subspaces of Γ are fundamental : $\Gamma_c = Cl\{\omega \in \Gamma \mid \omega$ is of class C^∞ and closed,

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i. e. $d\omega=0\}$, Cl being the closure taken in Γ. $\Gamma_e=Cl\{df\in\Gamma|f$ is a C^∞-function$\}$, $\Gamma_{e0}=Cl\{df|f$ is a C^∞-function with compact support on $R\}$. It is known [3] that $\Gamma_{e0}=\{dp|p$ is a Dirichlet potential on $R\}$. $\Gamma_h=\{\omega\in\Gamma|\omega$ is harmonic, i. e. $d\omega=d^*\omega=0\}$, $\Gamma_{he}=\left\{\omega\in\Gamma_h|\omega \text{ is exact, i. e. the period } \int_\gamma \omega \text{ of } \omega \text{ along every closed curve } \gamma \text{ on } R \text{ is zero}\right\}$, $\Gamma_{hse}=\left\{\omega\in\Gamma_h|\omega \text{ is semi-exact, i. e. } \int_\gamma \omega=0 \text{ along every dividing curve } \gamma \text{ on } R\right\}$, and Γ_{hm} $=$the orthogonal complement (in Γ_h) of $^*\Gamma_{hse}$, where $^*\Gamma_{hse}=\{\omega|{}^*\omega\in\Gamma_{hse}\}$. Every element of Γ_{hm} is also characterized as the limit of harmonic measure differentials. The following well-known *orthogonal decomposition* theorems will be used:

$$\Gamma=\Gamma_h+\Gamma_{e0}+{}^*\Gamma_{e0}$$
$$\Gamma_c=\Gamma_h+\Gamma_{e0},\ \Gamma_e=\Gamma_{he}+\Gamma_{e0},$$

(cf. [1], [5]).

2. By an *end* V of R we mean here a non-compact subregion of R whose relative boundary ∂V on R is a dividing curve. For studies of the boundary behavior of differentials in Γ we introduce the following notion.

DEFINITION *Let Γ_x be a subspace of Γ and V be an end of R. For $\omega\in\Gamma_x$ we call a differential $\tilde{\omega}$ in Γ_x the modification of ω (relative to V) if there exists a differential $\omega_0\in\Gamma_{e0}$ such that*

$$\tilde{\omega}=\begin{cases}\omega+\omega_0 & \textit{in } V'\subset V\\ \omega_0 & \textit{in } R-V,\end{cases} \tag{1}$$

where V' is a neighborhood of the ideal boundary to V. In the case $\omega\in\Gamma_x(V)$ (space Γ_x defined on V) we call $\tilde{\omega}$ on R satisfying the condition (1) *the extended modification of ω to R.*

Without loss of generality we may choose V' as an end contained in V so that $G=V-(V'\cup\partial V')$ is a ring domain. Then one can find a C^∞-function ρ on R such that

$$\rho=1 \text{ on } V', \text{ and } \rho=0 \text{ on } R-V.$$

If necessary, we shall take $\rho\in C^\infty$ so that its domain being non-constant is included in a smaller ring domain G' in G.

Note that in general ω_0 in (1) is not identically zero, for instance, if $\omega_0=0$ for $\omega\in\Gamma_x\subset\Gamma_h$, then ω would be identically zero. While, ω_0 does not influence the boundary behavior on V of ω, since, roughly speaking, ω_0 tends to zero towards ideal boundary.

PROPOSITION 1 *Let V be an end of Riemann surface R and $\Gamma_x(V)=\Gamma_{he}(V)$ or $\Gamma_{hse}(V)$. Then for every $\omega\in\Gamma_x(V)$ there exists the extended modification of ω to R.*

Proof Let $\omega\in\Gamma_{hse}(V)$. Since ∂V is a dividing curve, ω can be written as $\omega=du$ with a harmonic function u on ring domain G above. Define $\hat{\omega}$ by

$$\hat{\omega}=\begin{cases}\omega \text{ on } V'\\ d(\rho u) \text{ on } \bar{G}=G\cup\partial G\\ 0 \text{ on } R-V.\end{cases}$$

Then $\hat{\omega}\in\Gamma_c\cap C^\infty$, hence by orthogonal decomposition it is written as $\hat{\omega}=\tilde{\omega}+\omega_{e0}$ with $\tilde{\omega}$

$\in\Gamma_h$ and $\omega_{e0}=df\in\Gamma_{e0}\cap C^\infty$ because $\Gamma_{e0}\cap C^\infty\subset\Gamma_e\cap C^\infty=\Gamma_e^\infty(=\{df\in\Gamma\,|\,f\in C^\infty\})$. To see $\tilde{\omega}\in\Gamma_{hse}$, take the homology basis $\{\gamma_i\}_{i=1,2,\ldots}$ of dividing curves on R so that $\gamma_1=\partial V$ and $\gamma_i(i\geq 2)$ lie in $R-\bar{G}$. Then any dividing curve γ is homologous to $\sum_i c_i\gamma_i$ and $\int_\gamma\hat{\omega}=\sum_i c_i\int_{\gamma_i}\hat{\omega}=0$, obviously $\int_\gamma df=0$, consequently $\int_\gamma\tilde{\omega}=0$, i. e. $\tilde{\omega}\in\Gamma_{hse}$ and it is the extended modification of ω with $\omega_0=-\omega_{e0}$.

The case $\omega\in\Gamma_{he}(V)$ is treated similarly and we have only to show that $\int_\gamma\tilde{\omega}=0$ for every non-dividing curve γ on R. This is easily seen as one can choose homology basis of non-dividing curves so that they do not meet G.

PROPOSITION 2 *Let Γ_x be any one of the subspaces Γ_{he}, Γ_{hse}, Γ_{hm} and Γ_e, Γ_{e0}, then for every $\omega\in\Gamma_x$ there exists the modification $\tilde{\omega}$ of ω relative to an end V, and it is uniquely determined for the first three spaces. Furthermore, when $\omega=dw$ belongs to any one of the spaces Γ_{he}, Γ_{hm} and $\Gamma_{e0}\cap C^k(V)$ $(1\leq k\leq\infty)$ $\tilde{\omega}$ is given by $\tilde{\omega}=d(\rho w)+\omega_0$ for some $\omega_0\in\Gamma_{e0}$, where w is a function harmonic or of class $C^{k+1}(V)$ respectively.*

Proof The cases $\Gamma_x=\Gamma_{he}$ or Γ_{hse} are already proved, because the restriction to V of such $\omega\in\Gamma_x$ belongs to $\Gamma_x(V)$. Next,

1°) Let $\omega=dw\in\Gamma_{hm}$. Then there is a sequence $\{w_n\}$ of harmonic measures with respect to a canonical exhaustion $\{R_n\}$ ([1]) of R such that

$$||dw_n-dw||_{R_n}\to 0\ (n\to\infty),\ dw_n\in\Gamma_{hm}(R_n).$$

Consider the continuous extension $\hat{w}_n$ of w_n onto R such that $\hat{w}_n=w_n$ on R_n and $\hat{w}_n$ is constant on each component of $R-R_n$. Evidently $||d\hat{w}_n-dw||\to 0\ (n\to\infty)$ and we may suppose $\{\hat{w}_n\}$ converge to w uniformly on every compact set on R. By orthogonal decomposition of $d\hat{w}_n\in\Gamma_e$ we have

$$(2)\qquad d\hat{w}_n=du_n+dw_n^0,\ du_n\in\Gamma_{he},\ dw_n^0\in\Gamma_{e0}.$$

It is noted that $du_n\in\Gamma_{hm}$, because for any $\theta\in{}^*\Gamma_{hse}$ we have $(du_n,\theta)=(d\hat{w}_n-dw_n^0,\theta)=(d\hat{w}_n,\theta)=(dw_n,\theta)_{R_n}=0$ since ∂R_n consists of dividing curves hence $\theta\in{}^*\Gamma_{hse}(R_n)$. Next, define σ_n by

$$(3)\qquad \sigma_n=d(\rho\hat{w}_n)=d(\rho u_n)+d(\rho w_n^0)$$

and consider the orthogonal decomposition of $d(\rho u_n)\in\Gamma_e\cap C^\infty$:

$$(4)\qquad d(\rho u_n)=dv_n+dw_n^1,\ dv_n\in\Gamma_{he},\ dw_n^1\in\Gamma_{e0}\cap C^\infty.$$

One can prove $dv_n\in\Gamma_{hm}$ as before. It is also proved as follows. For $m>n$ let $u_{n,m}$ be the harmonic measure on R_m having the same boundary value as $\hat{w}_n|_{R_m}$, then $\lim_{m\to\infty}u_{n,m}=u_n$. Moreover let $v_{n,m}$ be the harmonic measure on R_m having the same boundary value as $\rho u_{n,m}$, then we find that $\lim_{m\to\infty}dv_{n,m}=dv_n\in\Gamma_{hm}$. Putting

$$(5)\qquad \hat{\omega}=\lim_{n\to\infty}\sigma_n(=d(\rho w)),$$

then $\hat{\omega}=\omega$ on V', $=0$ on $R-V$. Now we note that $\{dv_n\}$ and $\{dw_n^1\}$, $\{d(\rho w_n^0)\}$ are Cauchy sequences in Γ_{hm} and Γ_{e0} respectively. Obviously $\{d\hat{w}_n\}$ is a Cauchy sequence. Since Γ_{he} is orthogonal to Γ_{e0}, we have from (2) $||d\hat{w}_n-d\hat{w}_m||^2=||du_n-du_m||^2+||dw_n^0-dw_m^0||^2$, hence $\{du_n\}$ and $\{dw_n^0\}$ are Cauchy sequences. Moreover $\{\hat{w}_n\}$ and $\{u_n\}$, (hence $\{w_n^0\}$) converge uniformly on every compact set on R. Since $||d(\rho u_n)-d(\rho u_m)||^2=||d(\rho(u_n-u_m))||_G^2+||du_n-du_m||_{V-G}^2$ and $||d(\rho(u_n-u_m))||_G\leq||(u_n-u_m)d\rho||_G+||\rho(du_n-du_m)||_G$, we know that $\{d(\rho u_n)\}$ forms a Cauchy sequence, so do $\{dv_n\}$ and $\{dw_n^1\}$ from (4). Writting

$$\lim_{n\to\infty} dv_n=\tilde{\omega},\quad \lim_{n\to\infty} d(w_n^1+\rho w_n^0)=-\omega_0,$$

then $\tilde{\omega}\in\Gamma_{hm}$, $\omega_0\in\Gamma_{e0}$ and $\hat{\omega}=\tilde{\omega}-\omega_0$, i. e. $\tilde{\omega}$ is the modification of ω.

2°) Let $\omega\in\Gamma_{e0}$. By definition there is a sequence $\{df_n\}$ with C^∞-functions f_n with compact support, for which $||df_n-\omega||\to 0$ $(n\to\infty)$. Let u_n be harmonic functions on ring domain G having the same boundary values as $f_n|_G$. Then

(6) $$||df_n-df_m||_G\geq||du_n-du_m||_G \text{ (Dirichlet principle)}.$$

Let $\tilde{f}_n$ be functions on R defined so that $\tilde{f}_n=f_n$ on $V'\cup(R-V)$ and $=u_n$ on G. Then $\tilde{f}_n$ are continuous and $||d\tilde{f}_n||<\infty$. From (6) $\{du_n\}$ forms a Cauchy sequence in $\Gamma_{he}(G)$ and $\{u_n\}$ converge to a harmonic function u uniformly on every compact set on G, particularly on a smaller ring domain $G'(\bar{G}'\subset G)$ for which we may suppose $\rho=0$ or 1 on $G'-G$. Hence $\{d(\rho\tilde{f}_n)\}$ is a Cauchy sequence in Γ_{e0} and $\tilde{\omega}=\lim_{n\to\infty} d(\rho\tilde{f}_n)\in\Gamma_{e0}$ gives a modification of ω with $\omega_0=0$, because $\tilde{\omega}=\omega$ on V' as $||\omega-\tilde{\omega}||_{V'}\leq||\omega-df_n||_{V'}+||df_n-\tilde{\omega}||_{V'}\to 0$. The same reasoning is valid for $\omega\in\Gamma_e$.

Finally when $\omega\in\Gamma_{e0}\cap C^k(V)$, it can be written as $\omega=df_0$ with a Dirichlet potential f_0 on R which is of C^{k+1} on V. Then $\tilde{\omega}=d(\rho f_0)\in\Gamma_{e0}\cap C^k$, because ρf_0 is a Wiener potential and $||\tilde{\omega}||<\infty$ (cf. Hilfssatz 6.4., Bemerkung, s. 84 [3]).

§2. Locally canonical functions

3. A meromorphic differential ω on Riemann surface R is said to be *canonical* ([4], [5]) if ω has at most a finite number of poles and its real part is expressed in a neighborhood of the ideal boundary of R as

(7) $$\mathrm{Re}\,\omega=\sigma+\sigma_0,\quad \sigma\in\Gamma_{hm},\quad \sigma_0\in\Gamma_{e0}.$$

A (single-valued) meromorphic function f on R is said to be canonical if df is canonical differential. By localizing the condition (7) we defined in our former paper [6] the locally canonical functions, namely, for an end V of R a meromorphic function defined on $\bar{V}=V\cup\partial V$ is said to be *locally canonical* on V if Re df is written as (7) in a neighborhood $V'(\subset V)$ of the ideal boundary to V. Obviously a canonical function on R is locally canonical on every end of R. For a locally canonical function f on V we have the following formula

(8) $$||df||_V^2=i\int_{\partial V} f\overline{df},$$

where the integration is taken along ∂V in the positive direction with respect to V. This result was stated in [6] without detailed proof. Here we shall prove a slightly general formula, from which (8) follows.

PROPOSITION 3 *Let u be a harmonic function defined on end V which is written as $du=\sigma+\sigma_0$ with $\sigma\in\Gamma_{hm}$, $\sigma_0\in\Gamma_{e0}$ in a neighborhood $V'(\subset V)$ of the ideal boundary to V.*

Then for any harmonic differential ω *on* $\bar{V}$ *such that* $\|\omega\|_V<\infty$ and $*\omega$ *is semi-exact on* V, *we have*

(9) $$(du,\ \omega)_V=\int_{\partial V} u\ {*\omega}.$$

Proof Consider a canonical exhaustion $\{R_n\}$ of R, then by Green's formula on $V_n=V\cap R_n$

$$(du,\ \omega)_{V_n}=\int_{\partial V} u\ {*\omega}+\int_{\gamma_n} u\ {*\omega},$$

where $\gamma_n=\partial R_n\cap V$ is contained in end V' for large $n>N$. Hence it suffices to show that the last integral tends to 0 as $n\to\infty$. First one can write $u=v+f_0$ with $dv=\sigma$ and $df_0=\sigma_0(\in C^\infty)$ on V'. Applying Proposition 1 and 2 to V' instead of V we get the extended modification $\tilde{\theta}$ of $\theta={*\omega}\in\Gamma_{hse}(V)$ and the modifications $\tilde{\sigma}$, $\tilde{\sigma}_0$ of σ and σ_0, namely, $\tilde{\theta}\in\Gamma_{hse}$ and for some $\omega_0\in\Gamma_{e0}$, $\tilde{\theta}-\omega_0=\theta$ in V', $=0$ on $R-V$, and $\tilde{\sigma}\in\Gamma_{hm}$, $\tilde{\sigma}-\omega_1=d(\rho v)$ for some $\omega_1\in\Gamma_{e0}$ and $\tilde{\sigma}_0=d(\rho f_0)\in\Gamma_{e0}\cap C^\infty$. It follows that

$$\int_{\gamma_n} u{*\omega}=\int_{\partial R_n}(\rho v+\rho f_0)(\tilde{\theta}-\omega_0)$$
$$=-(\tilde{\sigma}-\omega_1+\tilde{\sigma}_0,\ {*\tilde{\theta}}-{*\omega_0})_{R_n}\to 0,\ n\to\infty,$$

because Γ_{hm} and Γ_{e0} are orthogonal to both $*\Gamma_{hse}$ and $*\Gamma_{e0}$.

COROLLARY 1 *Let* f *be a locally canonical function holomorphic on* $\bar{V}$, *and* g *a holomorphic function on* $\bar{V}$ *with finite norm* $\|dg\|_V$. *Then we have*

(10) $$(df,\ dg)_V=2i\int_{\partial V}\operatorname{Re} f\,\overline{dg}=i\int_{\partial V}(f+\bar{f})\overline{dg},$$

In particular, we have formula (8) *if* $f=g$.

Proof Since $(dh,\ dg)_V=2(\operatorname{Re} dh,\ dg)_V$ holds generally for any holomorphic function h with $\|dh\|_V<\infty$, it is sufficient to show that $(du,\ dg)_V=i\int_{\partial V} u\,\overline{dg}$ for $u=\operatorname{Re} f$. Now $dg=\omega+i{*\omega}$, where $\omega=d\operatorname{Re} g$ and $*\omega$ are both harmonic, exact together with their conjugates, hence Proposition 3 implies

$$(du,\ \omega)_V=\int_{\partial V} u{*\omega},\ (du,\ i{*\omega})_V=-i(du,\ {*\omega})_V=i\int_{\partial V} u\omega,$$

therefore $(du,\ dg)_V=i\int_{\partial V} u(\omega-i{*\omega})=i\int_{\partial V} u\,\overline{dg}$. Taking $f=g$ in(10), (8) is obtained as $\int_{\partial V}\bar{f}\,\overline{df}=\frac{1}{2}\overline{\int_{\partial V} df^2}=0.$

COROLLARY 2 *Let* F *be a canonical function on* R *with a finite number of poles* $\{p_j\}$ *and* h *a holomorphic function with finite norm* $\|dh\|$, *then*

(11) $$(dh,\ dF)=-2\pi\sum_j\operatorname*{Res}_{p_j} F dh,$$

where the left hand side is the Cauchy's principal value of integral. In the case where iF *is canonical,*

$$(dh,\ dF)=2\pi\sum_j\operatorname*{Res}_{p_j} F dh.$$

Proof Choose $\varepsilon>0$ so that disc ε-neighborhoods $U_j(\varepsilon)$ of p_j are disjoint each other and let $R_\varepsilon=R-\bigcup_j U_j(\varepsilon)$. It is also easy to find a finite number of ends $\{V_k\}$ of R such that $R-\bigcup_k V_k$ is a compact subregion containing $\bigcup_j U_j(\varepsilon)$. Then F is locally canonical on each V_k, hence by Green's formula and above Corollary we have easily

$$(dh,\ dF)_{R_\varepsilon}=\overline{(dF,\ dh)}_{R_\varepsilon}=i\sum_j\int_{\partial U_j(\varepsilon)}(F+\bar{F})dh$$
$$=-2\pi\sum_j \operatorname{Res}_{p_j} Fdh+O(\varepsilon).$$

Letting $\varepsilon\to 0$, (11) is obtained.

§3. Canonical differentials and functions

4. The existence of canonical functions is known from Riemann-Roch theorem on open Riemann surfaces [4], but we want to show here another version of existence theorem. For this aim we need the following fact, for which we shall give new proof (cf. [2], [8]).

LEMMA *Let ω be a semi-exact canonical differential on Riemann surface R of genus $g(<\infty)$, then*

$$\deg(\omega)\ \leq\ 2g-2.$$

Proof Denoting by $n_\omega(\alpha,\ R')$ the number (counted with multiplicities) of α-points ($\alpha=0,\ \infty$) of ω on subregion R' of R, then $\deg(\omega)=n_\omega(0,\ R)-P$, where $P=n_\omega(\infty,\ R)<\infty$. Take a canonical exhaustion $\{R_n\}$ of R, where ∂R_n consists of dividing analytic curves. First fix a subregion R_n which contains all poles of ω and is of genus g. It is known [4] that $\omega=df$ can be approximated by semi-exact canonical differentials $\omega_m=df_m$ on R_m which have the same singularities as ω and Re f_m take actually constants on ∂R_m, and hence $\omega-\omega_m$ converge to zero uniformly on every compact subset of R. Therefore by argument principle we see easily that

$$n_\omega(0,\ R_n)\leq n_{\omega_m}(0,\ R_n)\leq n_{\omega_m}(0,\ R_m),$$

provided $m(>n)$ is sufficiently large. Now choose a subregion R'_m of R_m so close to R_m that f_m is single-valued on every ring domains of $R_m-R'_m$ (note that ω_m is semi-exact) and ω_m has no zeros there. Then by Gauss-Bonnet theorem on R'_m

$$n_{\omega_m}(0,\ R_m)-P=\frac{1}{2\pi}\int_{\partial R'_m} d\arg df_m+\chi(R'_m),$$

where $\chi(R'_m)=2g+l_m-2$ is the Euler characteristic of R'_m(and of R_m), l_m being the number of contours $\{\gamma_i\}$ of R'_m. As Re f_m is constant on each contour of ∂R_m, the image $f_m(\partial R'_m)$ consists of closed curves, each of which turns around a vertical slit at least once in the negative direction (positive with respect to $f_m(R'_m)$), hence $(2\pi)^{-1}\int_{\partial R'_m} d\arg df_m\ \leq -l_m$. It follows that for any n

$$n_\omega(0,\ R_n)-P\ \leq\ 2g-2.$$

Letting $n\to\infty$, we have the conclusion.

5. Let $M(p^{-n})$ and $D(p^n)$ denote the real vector spaces of canonical functions and

semi-exact canonical differentials on R which are the multiple of divisor p^{-n} and p^n respectively. By Riemann-Roch theorem for open Riemann surface of genus $g(<\infty)$ we have

(12) $$\dim M(p^{-n})=\dim D(p^n)+2(n-g+1).$$

While for $n\geq 2g-1$, $\dim D(p^n)=0$ by Lemma, hence

(12)′ $$\dim M(p^{-n})=2(n-g+1),\ n\geq 2g-1.$$

In particular $\dim M(p^{-n})-\dim M(p^{-(n-1)})=2$, $n\geq 2g$, so that one can easily find linearly independent canonical functions f_1, f_2 with pole p of order n and a local parameter z at p ($p\leftrightarrow z=0$) for which they are *normalized* in the form

(13) $$f_1(z)=1/z^n+\cdots\cdots,\ f_2(z)=i/z^n+\cdots\cdots$$

Note that in the case $g=0$ we have only to take the semi-exact canonical differentials $\omega_i=df_i$ with above singularities, which always exist and become exact on R. Combining these facts with Theorems 1 and 4 in [6], the following theorem is obtained.

MAIN THEOREM *Let p be any point of open Riemann surface R of genus $g(<\infty)$ and $n\geq\max(2g, 1)$. Then on R there exist two canonical functions f_1, f_2 which have a pole only at p of order n and normalized there. In other words, f_1 and $f_0=-if_2$ give respectively the vertical and horizontal slit conformal mappings of R, more precisely, f_1 and f_0 take every complex values exactly n times except a set E of 2-dimensional measure zero, and each component of E is a vertical resp. horizontal slit (possibly a point) on the complex plane, further f_1, f_0 have a pole at p of order n and the normalized expansions of the form $z^{-n}(1+\cdots\cdots)$ in terms of a local parameter z at p.*

Remark Since $\dim M(p^{-(g+1)})\geq 4$, there is non-constant vertical or horizontal slit mapping of R with a pole p of order at most $g+1$. While, above theorem states the existence of a pair of normalized slit mappings with pole p of same given order $n\geq\max(2g, 1)$, which shows a complete generalization of classical parallel slits mapping theorem including Riemann's theorem of conformal mapping.

To get the canonical functions with a pole of lower order we need an analog of classical Weierstrass' gap theorem. Concerning this we give just a comment. From (12) $\dim D(p^{n-1})-2\leq\dim D(p^n)\leq\dim D(p^{n-1})$, and as $\dim D(p^0)=2g$ we have $\dim D(p^n)\geq 2(g-n)$, $0\leq n\leq g$, where the equality holds also for $n=1$ by (12). Note that

(14) $$\dim D(p^n)=2(g-n)\ \text{for}\ n=2,\cdots\cdots,g$$

if and only if there does not exist canonical functions with pole p of order at most g. Such a point is called non-Weierstrass point and the set E of those points is known to be dense in R, actually all points of R except a real analytic set belong to E [7].

Question: Does the case

(15) $$\dim D(p^n)=\dim D(p^{n-1})-1,\ \text{i. e.}\ \dim M(p^n)-\dim M(p^{-(n-1)})=1$$

occur for some n with $2\leq n\leq 2g-1$?

§4. Remarks to the case of infinite genus

6. Contrary to the case of finite genus, little is known about the existence of canonical

functions on Riemann surface R of infinite genus. Here we provide some remarks relevant to this problem in specific surfaces. According to Theorem 1 in [6], Riemann surface permitting a nonconstant canonical function is conformally equivalent to a finite sheeted covering surface over $\mathbb{C}$. The converse is not true as follows.

EXAMPLE 1. Let S be a two-sheeted covering surface over $\mathbb{C}$ with an infinite number of branch points $\{a_n\}$ which cluster only to ∞. Consider a disc D lying on the upper sheet, then $R=S-\bar{D}$ is of genus ∞ and does not permit canonical functions with a pole lying outside of D and its lift. Indeed, let f be such function, then f is bounded in neighborhood of ∞ (Th. 1, [6]) hence by well-known Myrberg's argument f takes the same value on two sheets and becomes a meromorphic function on $\mathbb{C}$, in particular, f is regular in D and transfers its boundary ∂D to a vertical segment, which is a contradiction.

While, there are Riemann surfaces of genus ∞ which permit canonical functions, namely,

EXAMPLE 2. Take a surface S above and delete from S a finite number of vertical slits not containing $\{a_n\}$ and let R be the resulting surface. Then for each n, $f(z)=1/(z-a_n)$ is lifted to canonical function on R with pole a_n of order 2.

EXAMPLE 3. Let G be a plane domain including ∞. By using two copies of G we make a two sheeted covering surface R with an infinite number of branch points $\{a_n\}\subset G$ which accumulate only to ∞. Then,

(i) for any point $p\in\{a_n\}$ and positive integer m there are vertical and horizontal slit mappings F_1 and F_2 having p a pole of order $2m$, where their expansions in local parameter t at p are of the form

$$t^{2m}+(\text{reg. function of } t).$$

Moreover there does not exist non-constant canonical function with pole $p\in\{a_n\}$ of odd order.

(ii) for any $p\notin\{a_n\}$, $\dim M(p^{-m})=2$, $m\geq 1$.

These facts imply that the case (15) does not occur for this surface.

Proof (i) There exist semi-exact canonical differentials ω_1, ω_2 on G having the singularity $d(1/(z-p)^m)$, $d(i/(z-p)^m)$ at p respectively. As G is planar, ω_1 and ω_2 are exact and their integrals f_1, f_2 are canonical functions on G. Then F_1, F_2, the lifts of f_1 and $-if_2$ onto R, have the property in (i). Next, canonical function f on R with a pole p takes the same values on two sheets and becomes a meromorphic function on G. Hence the order of pole at branch point p must be even.

(ii) For $p\notin\{a_n\}$, $M(p^{-m})$ always consists of constants, because its member never fail to have another pole at the lift of p.

7. REMARK Suppose a Riemann surface R permits the vertical and horizontal slit mapping F_1, F_2 with a pole p of order 2. This is the case, for instance any Riemann surface of genus 1 or above examples of genus $g(\leq\infty)$. Choosing a suitable local parameter z at p, they are written as

$$F_1(z)=z^{-2}+\sum_{n=0}^{\infty} a_n z^n,\quad F_2(z)=z^{-2}+kz^{-1}+\sum_{n=0}^{\infty} b_n z^n$$

at p. For any holomorphic function h on R with $\|dh\|<\infty$ which has the expansion $h(z)=\sum_{n=0}^{\infty} c_n z^n$ at p, we have by Corollary 2

$$(dh,\ dF_1)=-4\pi c_2,\quad (dh,\ dF_2)=4\pi c_2+2\pi k c_1.$$

Thus if we consider as usual the functions

$$\Psi=\frac{1}{2}(F_2-F_1),\quad \Phi=\frac{1}{2}(F_1+F_2)$$

we have $(dh,\ d\Psi)=4\pi c_2+\pi k c_1$, $(dh,\ d\Phi)=\pi k c_1$. Now it is interesting the case $k=0$. Then Ψ is holomorphic and $\|d\Psi\|<\infty$, hence taking $h=\Psi$

$$\|d\Psi\|^2=4\pi(b_2-a_2),$$

which implies b_2-a_2 is real, non-negative and $\mathrm{Re}\,a_2=\mathrm{Re}\,b_2$ if and only if $F_2=F_1+\text{const}$.

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